Field theoretic calculation of renormalized-viscosity, renormalized-resistivity, and energy fluxes of magnetohydrodynamic turbulence

Mahendra K. Verma *
Department of Physics, Indian Institute of Technology, Kanpur – 208016, INDIA
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A self-consistent renormalization (RG) scheme has been applied to nonhelical magnetohydrodynamic turbulence with zero cross helicity. Kolmogorov's 5/3 powerlaw has been shown to be a consistent solution for $d \geq d_c \approx 2.2$. For Kolmogorov's solution, both renormalized viscosity and resistivity are positive for the whole range of parameters. Various cascade rate and Kolmogorov's constant for MHD turbulence have been calculated by solving the flux equation to the first order in perturbation series. We find that the magnetic energy cascades forward. The Kolmogorov's constant for $d = 3$ does not vary significantly with r_A and is found to be close to the constant for fluid turbulence.

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The statistical theory of magnetohydrodynamic (MHD) turbulence is one of the important problems of current research. The quantities of interests in this area are energy spectrum, cascade rates, intermittency exponents etc. In this letter we analytically compute the renormalized-viscosity, renormalized-resistivity, and cascade rates using the field-theoretic techniques.

The incompressible MHD equation in Fourier space is given by

$$\left(-i\omega + \nu k^2\right) u_i(\hat{k}) = -\frac{i}{2} P^+_{ijm}(\mathbf{k}) \int d\hat{p} [u_j(\hat{p}) u_m(\hat{k}-\hat{p}) - b_j(\hat{p}) b_m(\hat{k}-\hat{p})] \tag{1}$$

$$\left(-i\omega + \lambda k^2\right) b_i(\hat{k}) = -i P^-_{ijm}(\mathbf{k}) \int d\hat{p} [u_j(\hat{p}) b_m(\hat{k}-\hat{p})] \tag{2}$$

$$k_i u_i(\mathbf{k}) = 0 \tag{3}$$

$$k_i b_i(\mathbf{k}) = 0 \tag{4}$$

where $\mathbf{u}$ and $\mathbf{b}$ are the velocity and magnetic field fluctuations respectively, ν and λ are the viscosity and the resistivity respectively, and d is the space dimension. Also,

$$P^+_{ijm}(\mathbf{k}) = k_j P_{im}(\mathbf{k}) + k_m P_{ij}(\mathbf{k}); \tag{5}$$

$$P_{im}(\mathbf{k}) = \delta_{im} - \frac{k_i k_m}{k^2}; \tag{6}$$

$$P^-_{ijm}(\mathbf{k}) = k_j \delta_{im} - k_m \delta_{ij}; \tag{7}$$

$$\hat{k} = (\mathbf{k}, \omega); \qquad d\hat{p} = d\mathbf{p} d\omega/(2\pi)^{d+1}. \tag{8}$$

The energy spectra for MHD, $E^u(k)$ and $E^b(k)$, are still under debate. Kraichnan [1] and Irosnikov [2] first gave phenomenology of steady-state, homogeneous, and isotropic MHD turbulence, and proposed that the spectra is proportional to $k^{-3/2}$. Later Marsch [3], Matthaeus and Zhou [4], and Zhou and Matthaeus [5] proposed an alternate phenomenology in which the energy spectra are proportional to $k^{-5/3}$, similar to Kolmogorov's spectrum for fluid turbulence. Current numerical [6–8] and theoretical [9–11] work support Kolmogorov-like phenomenology for MHD turbulence. In the present paper we show that Kolmogorov's spectrum ($\propto k^{-5/3}$) is a consistent solution of renormalization group (RG) equation of MHD turbulence.

Forster et al., DeDominicis and Martin, Fournier and Frisch, Yakhot and Orszag [12] applied RG technique to fluid turbulence. They considered external forcing and calculated renormalized parameters: viscosity, noise coefficient, and

*email: mkv@iitk.ac.in

vertex. McComb and his coworkers [13] instead applied a self-consistent RG procedure; here the energy spectrum was assumed to be Kolmogorov's powerlaw, and the renormalized viscosity was computed iteratively. For MHD turbulence, Fournier et al., Camargo and Tasso, and Liang and Diamond [14] employed RG technique in the similar lines as Forster et al. [12]. In this letter we will apply McComb's self-consistent technique to MHD turbulence. Earlier Verma [9] had done a self-consistent calculation and showed that the mean magnetic field gets renormalized, and the Kolmogorov's powerlaw is a consistent solution of MHD RG equation. In this letter we will carry out the renormalization of viscosity and resistivity.

For simplicity of the calculation we assume that the mean magnetic field is absent. This allows us to assume the turbulence to be isotropic to a reasonable approximation. In presence of mean magnetic field, turbulence become anisotropic; this issue has been studied by Sridhar and Goldreich [10] and Goldreich and Sridhar [11]. In addition to the above assumption, we also take cross helicity ($2\mathbf{u}\cdot\mathbf{b}$), magnetic helicity ($\mathbf{a}\cdot\mathbf{b}$), and kinetic helicity ($\mathbf{u}\cdot\omega$) to be zero, where $\mathbf{a}$ is magnetic vector potential, and ω is the vorticity.

In our RG procedure the wavenumber range (k_N, k_0) is divided logarithmically into N shells. We carry out the elimination of the first shell $k^> = (k_1, k_0)$ and obtain the modified MHD equation for $k^< = (k_N, k_1)$. This process is continued for all the shells. The shell elimination is performed by ensemble averaging over $k^>$ modes [12,14]. We assume that $u_i^>(\hat{k})$, and $b_i^>(\hat{k})$ have gaussian distributions with zero mean, while $u_i^<(\hat{k})$ and $b_i^<(\hat{k})$ are unaffected by the averaging process. In addition we take

$$\left\langle u_i^>(\hat{p}) u_j^>(\hat{q})\right\rangle = P_{ij}(\mathbf{p}) C^{uu}(\hat{p})\delta(\hat{p}+\hat{q}) \tag{9}$$
$$\left\langle b_i^>(\hat{p}) b_j^>(\hat{q})\right\rangle = P_{ij}(\mathbf{p}) C^{bb}(\hat{p})\delta(\hat{p}+\hat{q}) \tag{10}$$

Let us denote $\nu_{(n)}$ and $\lambda_{(n)}$ as the viscosity and resistivity after the elimination of the n shell. To first order of perturbation, we obtain

$$\begin{aligned}\left(-i\omega+\nu_{(n)}k^2+\delta\nu_{(n)}k^2\right) u_i^<(\hat{k}) = -\frac{i}{2}P^+_{ijm}(\mathbf{k})\int d\hat{p}[u_j^<(\hat{p})u_m^<(\hat{k}-\hat{p})\\ -b_j^<(\hat{p})b_m^<(\hat{k}-\hat{p})]\end{aligned} \tag{11}$$
$$\left(-i\omega+\lambda_{(n)}k^2+\delta\lambda_{(n)}k^2\right) b_i^<(\hat{k}) = -iP^-_{ijm}(\mathbf{k})\int d\hat{p}[u_j^<(\hat{p})b_m^<(\hat{k}-\hat{p})] \tag{12}$$

where

$$\begin{aligned}\delta\nu_{(n)}(k) = \frac{1}{(d-1)k^2}\int^{\Delta}_{\hat{p}+\hat{q}=\hat{k}} \frac{d\mathbf{p}}{(2\pi)^d}[S(k,p,q)\frac{C^{uu}(q)}{\nu_{(n)}(p)p^2+\nu_{(n)}(q)q^2}\\ -S_6(k,p,q)\frac{C^{bb}(q)}{\lambda_{(n)}(p)p^2+\lambda_{(n)}(q)q^2}]\end{aligned} \tag{13}$$
$$\begin{aligned}\delta\lambda_{(n)}(k) = \frac{1}{(d-1)k^2}\int^{\Delta}_{\hat{p}+\hat{q}=\hat{k}} \frac{d\mathbf{p}}{(2\pi)^d}[-S_8(k,p,q)\frac{C^{bb}(q)}{\nu_{(n)}(p)p^2+\lambda_{(n)}(q)q^2}\\ +S_9(k,p,q)\frac{C^{uu}(q)}{\lambda_{(n)}(p)p^2+\nu_{(n)}(q)q^2}]\end{aligned} \tag{14}$$

with $S_i(k,p,q)$s as functions of k, p, and q. Hence, after the elimination of the $(n+1)$th shell, the effective viscosity and resistivity will be $(\nu,\lambda)_{(n+1)}(k) = (\nu,\lambda)_{(n)}(k) + \delta(\nu,\lambda)_{(n)}(k)$.

We solve the above equations iteratively. To simplify, we substitute $C(k)$ in the Eqs. (13, 14) by one dimensional energy spectrum $E(k)$

$$C^{uu,bb}(k) = \frac{2(2\pi)^d}{S_d(d-1)}k^{-(d-1)}E^{u,b}(k) \tag{15}$$

where S_d is the surface area of d dimensional spheres. We assume that $E^u(k)$ and $E^b(k)$ follow

$$E^u(k) = K^u\Pi^{2/3}k^{-5/3}; \qquad E^b(k) = E^u(k)/r_A \tag{16}$$

Regarding $\nu_{(n)}$ and $\lambda_{(n)}$, we attempt the following form of solution

$$(\nu,\lambda)_{(n)}(k_n k') = (K^u)^{1/2}\Pi^{1/3}k_n^{-4/3}(\nu,\lambda)^*_{(n)}(k') \tag{17}$$

with $k = k_{n+1}k'(k' < 1)$ with the expectation that $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$ are universal functions for large n. We numerically solve for $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$. Our calculations reveal that the solutions of $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$ converge for all $d > d_c \approx 2.2$. From this observation we can conclude that Kolmogorov's energy spectrum ($E(k) \propto k^{-5/3}$) is a consistent solution of the RG equations. Meanwhile, Kraichnan's 3/2 energy spectrum and $\nu k^2 = \lambda k^2 \propto k B_0$, where B_0 (a constant) is the magnetic field of the large eddies, do not satisfy the renormalization group equations [Eqs. (13, 14)]. Hence $E(k) \propto k^{-3/2}$ is not a consistent solution of RG equations. Our result regarding the nonexistence of stable RG fixed point for $d = 2$ is consistent with the RG calculation of Liang and Diamond [14]. Refer to Fig. (1) for illustration of $\nu^*_{(n)}(k')$ and $\lambda^*_{(n)}(k')$ for $d = 3$ and $r_A = 1$.

The values of renormalized parameters for $d = 3$ and various r_A are shown in Table I. For large r_A, the asymptotic ν^* is close to the corresponding value for fluid turbulence, but the asymptotic λ^* is also comparable to $\nu*$. This implies that in fluid dominated regime, there is a significant magnetic energy flux in addition to the usual Kolmogorov's flux in fluid modes. As r_A is decreased, ν^* increases but λ^* decreases. This trend is seen till $r_A \approx 0.25$, where the RG fixed point with nonzero ν^* and λ^* becomes unstable, and the trivial RG fixed point with $\nu^* = \lambda^* = 0$ becomes stable. This result suggests an absence of turbulence for r_A below 0.25. This is consistent with the fact that MHD equations become linear in the $r_A \rightarrow 0$ (fully magnetic) limit.

We can proceed further and compute various cascade rates and Kolmogorov's constant for MHD using the renormalized parameters computed above. To compute these quantities we resort to the energy equations, which are [15,16]

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) C^{uu}(\mathbf{k}, t) = \frac{1}{(d-1)(2\pi)^d \delta(\mathbf{k}+\mathbf{k'})} \int_{\mathbf{k'}+\mathbf{p}+\mathbf{q}=\mathbf{0}} \frac{d\mathbf{p}}{(2\pi)^d} [S^{uu}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) + S^{uu}(\mathbf{k'}|\mathbf{q}|\mathbf{p}) + S^{ub}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) + S^{ub}(\mathbf{k'}|\mathbf{q}|\mathbf{p})] \quad (18)$$

$$\left(\frac{\partial}{\partial t} + 2\lambda k^2\right) C^{bb}(\mathbf{k}, t) = \frac{1}{(d-1)(2\pi)^d \delta(\mathbf{k}+\mathbf{k'})} \int_{\mathbf{k'}+\mathbf{p}+\mathbf{q}=\mathbf{0}} \frac{d\mathbf{p}}{(2\pi)^d} [S^{bu}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) + S^{bu}(\mathbf{k'}|\mathbf{q}|\mathbf{p}) + S^{bb}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) + S^{bb}(\mathbf{k'}|\mathbf{q}|\mathbf{p})] \quad (19)$$

where

$$S^{uu}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k'}.\mathbf{u}(\mathbf{q})][\mathbf{u}(\mathbf{k'}).\mathbf{u}(\mathbf{p})]\right), \quad (20)$$

$$S^{bb}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) = -\Im\left([\mathbf{k'}.\mathbf{u}(\mathbf{q})][\mathbf{b}(\mathbf{k'}).\mathbf{b}(\mathbf{p})]\right), \quad (21)$$

$$S^{ub}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) = \Im\left([\mathbf{k'}.\mathbf{b}(\mathbf{q})][\mathbf{u}(\mathbf{k'}).\mathbf{b}(\mathbf{p})]\right), \quad (22)$$

$$S^{bu}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) = -S^{ub}(\mathbf{p}|\mathbf{k'}|\mathbf{q}) \quad (23)$$

Here $\Im$ stands for the imaginary part of the argument, and the above integrals have constraints that $\mathbf{k'} + \mathbf{p} + \mathbf{q} = \mathbf{0}$ ($\mathbf{k} = -\mathbf{k'}$). The energy equations in the above form have been written by Dar et al. [16], who interpret the terms $S(\mathbf{k}|\mathbf{p}|\mathbf{q})$ as energy transfer rate from mode $\mathbf{p}$ (second argument of S) to $\mathbf{k}$ (first argument of S) with mode $\mathbf{q}$ (third argument of S) acting as a mediator. This interpretation of energy transfer due to Dar et al. [16] is consistent with the earlier formalism.

We can derive an expression for the energy transfer rate or energy flux from a wavenumber sphere using $S(\mathbf{k'}|\mathbf{p}|\mathbf{q})$. The formula for the energy flux from inside of the X-sphere ($X <$) to outside of the Y-sphere ($Y >$) is

$$\Pi^{X<}_{Y>}(k_0) = \int_{k>k_0} \frac{d\mathbf{k}}{(2\pi)^d} \int_{p<k_0} \frac{d\mathbf{p}}{(2\pi)^d} \left\langle S^{YX}(\mathbf{k'}|\mathbf{p}|\mathbf{q}) \right\rangle \quad (24)$$

where X and Y stand for u or b. In our study we assume that the kinetic energy is forced at small wavenumbers, and the turbulence is steady. We calculate the above fluxes analytically to the leading order in perturbation series using the same procedure as that of Leslie [17]. The flux is calculated using Eq. (24) by taking ensemble average of S^{YX}. The expression for $\left\langle S^{bb}(k|p|q) \right\rangle$ is

$$\begin{aligned} \left\langle S^{bb}(k|p|q) \right\rangle = \int_{-\infty}^{t} dt' [& T_4(k,p,q) G^{bb}(k, t-t') C^{bb}(p, t-t') C^{uu}(q, t-t') \\ & + T_8(k,p,q) G^{bb}(p, t-t') C^{bb}(k, t-t') C^{uu}(q, t-t') \\ & + T_{10}(k,p,q) G^{uu}(q, t-t') C^{bb}(k, t-t') C^{bb}(p, t-t')] \end{aligned} \quad (25)$$

where $T_i(k,p,q)$ are functions of wavevectors $k,p,$ and q. The expressions for other transfer rates $\langle S^{uu}(k|p|q)\rangle$, $\left\langle S^{ub}(k|p|q)\right\rangle$, and $\left\langle S^{bu}(k|p|q)\right\rangle$ look similar. In the above formulas we substitute Kolmogorov's spectrum [Eqs. (16)] for the energy spectrum, and the following expression for the effective viscosity and resistivity

$$(\nu,\lambda)(k) = (K^u)^{1/2}\Pi^{1/3}k^{-4/3}(\nu^*,\lambda^*) \qquad \text{for } k \geq k_n \tag{26}$$

Following the same procedure as Leslie [17], we obtain the following nondimensional form of the equations

$$\frac{\Pi_{Y>}^{X<}}{\Pi} = \frac{4S_{d-1}}{(d-1)^2 S_d}(K^u)^{3/2}\int_0^1 dv \ln(1/v)\int_{1-v}^{1+v} dw (vw)^{d-2}(\sin\alpha)^{d-3}F_{Y>}^{X<} \tag{27}$$

where the integrals $F_{Y>}^{X<}$ are function of v, w, ν^*, and λ^*. After a bit of manipulation we can obtain $\Pi_{Y>}^{X<}/\Pi$ and the constant K^u. In addition we can also obtain the Kolmogrov's constant K for total energy

$$E(k) = K\Pi^{2/3}k^{-5/3} \tag{28}$$

using $K = K^u(1+r_A^{-1})$. The values of $\Pi_{Y>}^{X<}/\Pi$ and K for $d=3$ and various r_A are listed in Table I.

The entries in the Table I show that the cascade rates $\Pi_{b>}^{u<}$, $\Pi_{b>}^{b<}$, $\Pi_{b<}^{u<}$, $\Pi_{u>}^{b<}$are approximately of the same order for r_A between 0.5 and 1, but the flux $\Pi_{u>}^{u<}$is rather small. The sign of $\Pi_{b>}^{b<}$is positive, indicating that the ME cascades forward, that is from large length-scales to small length-scales. The magnetic energy thus appearing at small length-scales will be lost due to resistive dissipation, and the large-scale magnetic field is maintained by the $\Pi_{b<}^{u<}$flux. The Kolmogorov's constant K is approximately constant and is close to 1.6, same as that for fluid turbulence ($r_A = \infty$), all r_A greater than 0.5.

To summarize, we employed a self-consistent RG scheme to MHD turbulence and found that Kolmogorov's 5/3 powerlaw is a consistent solution of RG equations for $d \geq d_c \approx 2.2$. For Kolmogorov's solution, the renormalized viscosity and resistivity have been calculated, and they are found to be positive. For $d=3$, variation of ν^* and λ^* with r_A shows some interesting features. For large r_A, ν^* is same as that for fluid turbulence, but λ^* is also nonzero, in fact larger than ν^*. As r_A is decreased, ν^* increases but λ^* decreases until $r_A \approx 0.25$ at which value turbulence disappears.

Using the flux equations we have obtained various fluxes and Kolmogorov's constant K. For $d=3$, K does not vary significantly with the variation of r_A, and it is close to K for fluid turbulence. We find that the cascade rate from magnetic-sphere to outside magnetic-sphere ($\Pi_{b>}^{b<}$) is positive, a result consistent with the numerical results of Dar et al. [16].

In this paper we have restricted ourselves to nonhelical turbulence. Helical MHD turbulence is very important specially in the light of enhancement of magnetic energy (dynamo). However, the physics of helical turbulence is more complex with the appearance of inverse cascade of magnetic helicity etc. The field-theoretic analysis for this case will be taken up later. Recent studies show that the mean magnetic field has a strong effect on energy spectrum, and it induces anisotropy. A full-fledge field theory calculation in the presence of mean magnetic field is also necessary for a clearer picture of MD turbulence

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TABLE I. The values of $\nu^*, \lambda^*, \nu^{uu*}, \nu^{ub*}, \lambda^{bu*}, \lambda^{bb*}$ for for various r_A when $\mathbf{d=3}$ and $\sigma_c = 0$

r_A	ν^*	λ^*	$\Pi^{u<}_{u>}/\Pi$	$\Pi^{u<}_{b>}/\Pi$	$\Pi^{b<}_{u>}/\Pi$	$\Pi^{b<}_{b>}/\Pi$	$\Pi^{u<}_{b<}/\Pi$	K
∞	0.38	$--$	1	$--$	$--$	$--$	$--$	1.53
5000	0.36	0.85	1	3.5E-4	-1.05E-4	2.4E-4	1.3E-4	1.51
5	0.47	0.82	0.61	0.26	-0.050	0.19	0.13	1.51
1	1.00	0.69	0.12	0.39	0.12	0.37	0.49	1.50
0.5	2.11	0.50	0.037	0.33	0.33	0.30	0.63	1.65
0.3	11.0	0.14	0.011	0.36	0.42	0.21	0.63	3.26
0.2	$--$	$--$	$--$	$--$	$--$	$--$	$--$	$--$

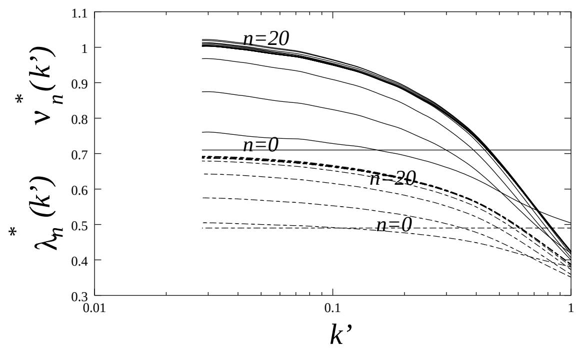

FIG. 1. Plot of $\nu^*(k')$ (solid lines) and $\lambda^*(k')$ (dashed lines) vs. k' for $d = 3$ and $\sigma_c = 0, r_A = 1$. Values at various iterations are shown by different curves.